QUANTITATIVE METHODS BASIC CONCEPTS

TIME VALUE OF MONEY

Overview —

- õ **Los a** Interpret interest rates as required rates of return, discount rates, or opportunity costs;
- õ **Los b** Explain an interest rate as the sum of a real risk-free rate, and premiums that compensate investors against inflation, liquidity, maturity and default risks;
- õ **Los c** Calculate and interpret the effective annual rate, given the stated annual interest rate and the frequency of compounding;
- õ **Los d** Solve time value of money problems for different frequencies of compounding;
- õ **Los e** Calculate and interpret the future value (FV) and present value (PV) of a single sum of money, an ordinary annuity, an annuity due, a perpetuity (PV only), and a series of unequal cash flows;
- õ **Los f** Demonstrate the use of a time line in modeling and solving time value of money problems;

Illustration

Example 1

The Future Value of a Lump Sum with Interim Cash Reinvested at the Same Rate

You are the lucky winner of your state's lottery of $5 million after taxes. You invest your winnings in a five-year certificate of deposit (CD) at a local financial institution. The CD promises to pay 7 percent per year compounded annually. This institution also lets you reinvest the interest at that rate for the duration of the CD. How much will you have at the end of five years if your money remains invested at 7 percent for five years with no withdrawals ?

Solution:

To solve this problem, compute the future value of the $5 million investment using the following values in Equation 2:

PV=$5,000,000 r=7% =0.07 N=5 FV_N $=PV(1+r)^N$ $=\$5,000,000(1.07)^5$ =$5,000,000(1.402552) =$7,012,758.65	PV=$5,000,000 r=7% =0.07 N=5 FV_N $= PV (1+r)^N$ $=\$5,000,000(1.07)^5$ =$5,000,000(1.402552) =$7,012,758.65

At the end of five years, you will have $7,012,758.65 if your money remains invested at 7 percent with no withdrawals.

Example 2

The Future Value of a Lump Sum with No Interim Cash

An institution offers you the following terms for a contract: For an investment of ¥2,500,000, the institution promises to pay you a lump sum six years from now at an 8 percent annual interest rate. What future amount can you expect ?

Solution:

Use the following data in Equation 2 to find the future value:

PV=¥2,500,000 r=8% = 0.08 N=6 FV_N $= PV(1+r)^N$ = ¥2,500,000$(1.08)^6$ = ¥2,500,000(1.586874) = ¥3,967,186	PV=¥2,500,000 r=8% = 0.08 N=6 FV_N $= PV(1+r)^N$ = ¥2,500,000$(1.08)^6$ = ¥2,500,000(1.586874) = ¥3,967,186

You can expect to receive ¥3,967,186 six years from now.

Example 3

The Future Value of a Lump Sum

A pension fund manager estimates that his corporate sponsor will make a $10 million contribution five years from now. The rate of return on plan assets has been estimated at 9 percent per year. The pension fund manager wants to calculate the future value of this contribution 15 years from now, which is the date at which the funds will be distributed to retirees. What is that future value?

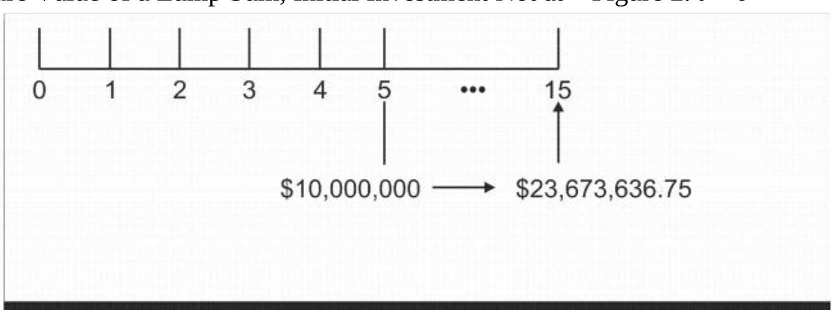
Solution:

By positioning the initial investment, PV, at t = 5, we can calculate the future value of the contribution using the following data in Equation 2:

PV =$10million r = 9% = 0.09 N = 10 FV_N = $PV(1+r)^N$ = $10,000,000 $(1.09)^{10}$ = $10,000,000 (2.367364) = $23,673,636.75	PV =$10million r = 9% = 0.09 N = 10 FV_N = $PV(1+r)^N$ = $10,000,000 $(1.09)^{10}$ = $10,000,000 (2.367364) = $23,673,636.75

This problem looks much like the previous two, but it differs in one important respect: its timing. From the standpoint of today (t = 0), the future amount of $23,673,636.75 is 15 years into the future. Although the future value is 10 years from its present value, the present value of $10 million will not be received for another five years.

Figure 2.The Future Value of a Lump Sum, Initial Investment Not at Figure 2. t = 0



As Figure 2 shows, we have followed the convention of indexing today as t = 0 and indexing subsequent times by adding 1 for each period. The additional contribution of $10 million is to be received in five years, so it is indexed as t = 5 and appears as such in the figure. The future value of the investment in 10 years is then indexed at t = 15; that is, 10 years following the receipt of the $10 million contribution at t = 5. Time lines like this one can be extremely useful when dealing with more-complicated problems, especially those involving more than one cash flow.

Example 4

The Future Value of a Lump Sum with Quarterly Compounding

Continuing with the CD example, suppose your bank offers you a CD with a two-year maturity, a stated annual interest rate of 8 percent compounded quarterly, and a feature allowing reinvestment of the interest at the same interest rate. You decide to invest $10,000. What will the CD be worth at maturity?

Solution:

Compute the future value with Equation 3 as follows:

$PV = \$10,000$ $r_S = 8\% = 0.08$ $m = 4$ $r_S / m = 0.008 / 4 = 0.02$ $N = 2$ mN $= PV\,(1 + \frac{r_S}{m})^{mN}$ $= \$10,000(1.02)^8$ $= \$10,000(1.171659)$ $= \$11.716.59$	$PV = \$10,000$ $r_S = 8\% = 0.08$ $m = 4$ $r_S / m = 0.008 / 4 = 0.02$ $N = 2$ mN $= PV\,(1 + \frac{r_S}{m})^{mN}$ $= \$10,000(1.02)^8$ $= \$10,000(1.171659)$ $= \$11.716.59$

At maturity, the CD Will be worth $11,716.59.

Example 5

The Future Value of a Lump Sum with Monthly Compounding

An Australian bank offers to pay you 6 percent compounded monthly. You decide to invest A$1 million for one year. What is the future value of your investment if interest payments are reinvested at 6 percent ?

Solution :

Use Equation 3 to find the future value of the one-year investment as follows :

PV = A$ 1,000,000	PV = A$1,000,000
r_S = 6% = 0.06	r_S = 6% = 0.06
m = 12	m = 12
r_S / m = 0.06 / 12 = 0.0050	r_S / m = 0.06 / 12 = 0.0050
N = 1	N = 1
mN = 12(1) = 12 interest periods	mN = 12(1) = 12 interest periods
$FV_N = PV = \left(1 + \frac{r_5}{m}\right)^{mN}$ $= A\$,000,00\left(1.005\right)^{12}$ $= A\$1,000,000\left(1.06178\right)$ $= A\$1,061,67781$	$FV_N = PV = \left(1 + \frac{r_5}{m}\right)^{mN}$ $= A\$,000,00\left(1.005\right)^{12}$ $= A\$1,000,000\left(1.06178\right)$ $= A\$1,061,67781$

If you had been paid 6 percent with annual compounding, the future amount would be only A$1,000,000(1.06) = A$1,060,000 instead of A$1,061,677.81 with monthly compounding.

Example 6

The Future Value of a Lump Sum with Continuous Compounding

Suppose a $10,000 investment will earn 8 percent compounded continuously for two years. We can compute the future value with Equation 4 as follows:

$$
\begin{aligned}
PV &= ¥2,500,000 \\
r &= 8\% = 0.08 \\
N &= 6 \\
FV_N &= PV(1 + r)^N \\
&= ¥2,500,000(1.08)^6 \\
&= ¥2,500,000(1.586874) \\
&= ¥3,967,186
\end{aligned}
$$

$$
\begin{aligned}
PV &= \$10.000 \\
r_S &= 8\% = 0.08 \\
N &= 2 \\
FV_N &= PVe^{rsN} \\
&= \$10,000e^{0.08}(2) \\
&= \$10,000(1.173511) \\
&= \$11.735.11
\end{aligned}
$$

With the same interest rate but using continuous compounding, the $10,000 investment will grow to $11,735.11 in two years, compared with $11,716.59 using quarterly compounding as shown in Example 4.

Example 7

The Future Value of an Annuity

Suppose your company's defined contribution retirement plan allows you to invest up to €20,000 per year. You plan to invest €20,000 per year in a stock index fund for the next 30 years. Historically, this fund has earned 9 percent per year on average. Assuming that you actually earn 9 percent a year, how much money will you have available for retirement after making the last payment?

Solution:

Use Equation 7 to find the future amount:

A = €20,000
r = 9% = 0.09
N = 30
FV annuity factor = (1+*r*)*N*−1*r*=(1.09)30−10.09=136.307539

$$\frac{(1 + r)^N - 1}{r} = \frac{(1.09)^{30} - 1}{0.09} = 136.307539$$

FV_N = €20,000(136.307539)
= €2,726,150.77

Assuming the fund continues to earn an average of 9 percent per year, you will have €2,726,150.77 available at retirement.

Example 8

The Present Value of a Lump Sum

An insurance company has issued a Guaranteed Investment Contract (GIC) that promises to pay $100,000 in six years with an 8 percent return rate. What amount of money must the insurer invest today at 8 percent for six years to make the promised payment ?

Solution:

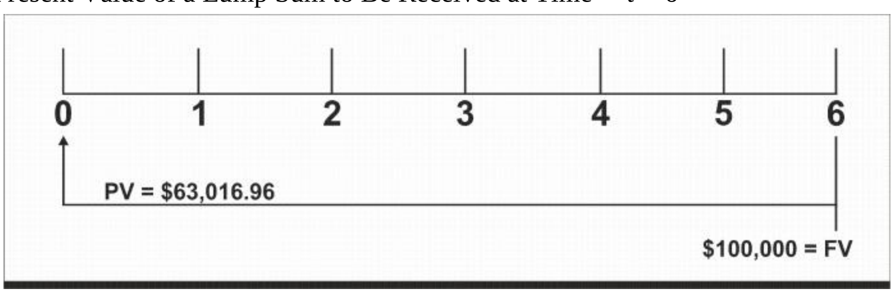
We can use Equation 8 to find the present value using the following data:

FV *N* = $100,000 *r* = 8% = 0.08*N* = 6PV = FV*N*(1+*r*)−*N*= $100,000[1(1.08)6] =$100,000(0.6301696) =$63,016.96

$$FV_N = \$100,000$$
$$r = 8\% = 0.08$$
$$N = 6$$
$$PV = FV_N(1+r)^{-N}$$
$$= \$100,000\left[\frac{1}{(1.08)^6}\right]$$
$$= \$100,000(0.6301696)$$
$$= \$63,016.96$$

We can say that $63,016.96 today, with an interest rate of 8 percent, is equivalent to $100,000 to be received in six years. Discounting the $100,000 makes a future $100,000 equivalent to $63,016.96 when allowance is made for the time value of money. As the time line in Figure 4 shows, the $100,000 has been discounted six full periods.

Figure 4. The Present Value of a Lump Sum to Be Received at Time $t = 6$



Example 9

The Projected Present Value of a More Distant Future Lump Sum

Suppose you own a liquid financial asset that will pay you $100,000 in 10 years from today. Your daughter plans to attend college four years from today, and you want to know what the asset's present value will be at that time. Given an 8 percent discount rate, what will the asset be worth four years from today?

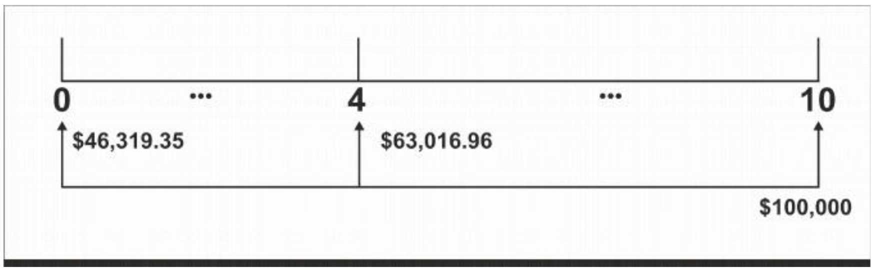
Solution:

The value of the asset is the present value of the asset's promised payment. At $t = 4$, the cash payment will be received six years later. With this information, you can solve for the value four years from today using Equation 8:

FV*N* = $100,000 *r* = 8% = 0.08*N* = 6PV = FV*N*(1+*r*)−*N* = $100,0001(1.08)6 =$100,000(0.6301696) =$63,016.96

$$FV_N = \$100,000$$
$$r = 8\% = 0.08$$
$$N = 6$$
$$PV = FV_N(1 + r)^{-N}$$
$$= \$100,000\left[\frac{1}{(1.08)^6}\right]$$
$$= \$100,000(0.6301696)$$
$$= \$63,016.96$$

Figure 5. The Relationship between Present Value and Future Value



The time line in Figure 5 shows the future payment of $100,000 that is to be received at t = 10. The time line also shows the values at t = 4 and at t = 0. Relative to the payment at t = 10, the amount at t = 4 is a projected present value, while the amount at t = 0 is the present value (as of today).

Example 10

The Present Value of a Lump Sum with Monthly Compounding

The manager of a Canadian pension fund knows that the fund must make a lump-sum payment of C$5 million 10 years from now. She wants to invest an amount today in a GIC so that it will grow to the required amount. The current interest rate on GICs is 6 percent a year, compounded monthly. How much should she invest today in the GIC?

Solution:

Use Equation 9 to find the required present value:

$$FV_N = C\$5,000,000 \quad r_s = 6\% = 0.06 \quad m = 12 \quad r_s/m = 0.06/12 = 0.005 \quad N = 10 \quad mN = 12(10) = 120$$
$$PV = FV_N\left(1 + \frac{r_s}{m}\right)^{-mN} = C\$5,000,000(1.005)^{-120} = C\$5,000,000(0.549633) = C\$2,748,163.67$$

$FV_N = C\$5,000,000$

$r_S = 6\% = 0.06\%$

$m = 12$

$r_S / m = 0.06 / 12 = 0.005$

$N = 10$

$mN = 12(10) = 120$

$$PV = FV_N (1 + \frac{r_S}{m})^{-mN}$$

$$= C\$5,000,000(1.005)^{-120}$$

$$= C\$5,000,000(0.549633)$$

$$= C\$2,748,163.67$$

In applying Equation 9, we use the periodic rate (in this case, the monthly rate) and the appropriate number of periods with monthly compounding (in this case, 10 years of monthly compounding, or 120 periods).

Example 11

The Present Value of an Ordinary Annuity

Suppose you are considering purchasing a financial asset that promises to pay €1,000 per year for five years, with the first payment one year from now. The required rate of return is 12 percent per year. How much should you pay for this asset?

Solution:

To find the value of the financial asset, use the formula for the present value of an ordinary annuity given in Equation 11 with the following data:

A = €1,000

r = 12% = 0.12

N = 5

$$PV = A[1 - 1(1 + r)Nr]\, A\, 1 - \left[\frac{1 - \frac{1}{(1+r)^N}}{r}\right]$$

$$= €1,1000 = [1 - 1(1.12)5.0.12] \left[\frac{1 - \frac{1}{(1+r)^5}}{0.12}\right]$$

$$= €1,000(3.604776)$$

$$= €3,604.78$$

The series of cash flows of €1,000 per year for five years is currently worth €3,604.78 when discounted at 12 percent.

Example 12

An Annuity Due as the Present Value of an Immediate Cash Flow Plus an Ordinary Annuity

You are retiring today and must choose to take your retirement benefits either as a lump sum or as an annuity. Your company's benefits officer presents you with two alternatives: an immediate lump sum of $2 million or an annuity with 20 payments of $200,000 a year with the first payment starting today. The interest rate at your bank is 7 percent per year compounded annually. Which option has the greater present value? (Ignore any tax differences between the two options.)

Solution:

To compare the two options, find the present value of each at time $t = 0$ and choose the one with the larger value. The first option's present value is $2 million, already expressed in today's dollars. The second option is an annuity due. Because the first payment occurs at $t = 0$, you can separate the annuity benefits into two pieces: an immediate $200,000 to be paid today ($t = 0$) and an ordinary annuity of $200,000 per year for 19 years. To value this option, you need to find the present value of the ordinary annuity using Equation 11 and then add $200,000 to it.

A=$200,000*N*=19*r*=7%=0.07PV=*A*[1−1(1+*r*)*Nr]* =$200,000[1−1(1.07)190.07]=$200,000(10.335595)=$2,067,119.05

A= $200.000

$N = 19$

r = 7% = 0.07

$$PV = A\left[\frac{1 - \frac{1}{(1+r)^N}}{r}\right]$$

$$= \$200,000\left[\frac{1 - \frac{1}{(1.07)^{19}}}{0.07}\right]$$

$$= \$200,000(10.3355595)$$

$$= \$2,067.119.05$$

The 19 payments of $200,000 have a present value of $2,067,119.05. Adding the initial payment of $200,000 to $2,067,119.05, we find that the total value of the annuity option is $2,267,119.05. The present value of the annuity is greater than the lump sum alternative of $2 million.

Example 13

The Projected Present Value of an Ordinary Annuity

A German pension fund manager anticipates that benefits of €1 million per year must be paid to retirees. Retirements will not occur until 10 years from now at time $t = 10$. Once benefits begin to be paid, they will extend until $t = 39$ for a total of 30 payments. What is the present value of the pension liability if the appropriate annual discount rate for plan liabilities is 5 percent compounded annually?

Solution:

This problem involves an annuity with the first payment at $t = 10$. From the perspective of $t = 9$, we have an ordinary annuity with 30 payments. We can compute the present value of this annuity with Equation 11 and then look at it on a time line.

A = €1,000,000

$r = 5\% = 0.05$

$N = 30$

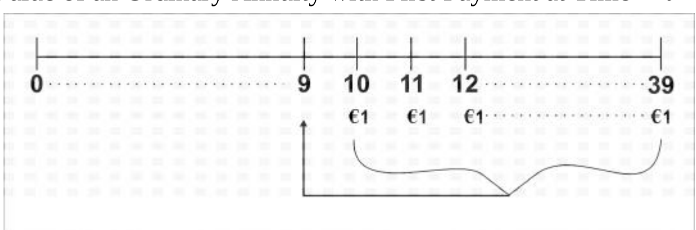
$$\text{PV} = \text{A}[1 - 1(1 + \text{r})\text{Nr}]\ \text{A}\ 1 - \left[\frac{1 - \frac{1}{(1+\text{r})^{\text{N}}}}{\text{r}}\right]$$

$$= €1,1000,000 = [1 - 1(1.05)30.0.05]\left[\frac{1 - \frac{1}{(1.05)^{30}}}{0.05}\right]$$

$$= €1,000,000(15.372451)$$

$$= €15,372,451,03$$

Figure 7. The Present Value of an Ordinary Annuity with First Payment at Time $t = 10$ (in Millions)



On the time line, we have shown the pension payments of €1 million extending from $t = 10$ to $t = 39$. The bracket and arrow indicate the process of finding the present value of the annuity, discounted back to $t = 9$. The present value of the pension benefits as of $t = 9$ is €15,372,451.03. The problem is to find the present value today (at $t = 0$).

Now we can rely on the equivalence of present value and future value. As Figure 7 shows, we can view the amount at $t = 9$ as a future value from the vantage point of $t = 0$. We compute the present value of the amount at $t = 9$ as follows:

FV_N = €15,372,451.03 (the present value at $t = 9$)

$N = 9$

$r = 5\% = 0.05$

$\text{PV} = \text{FV}_N(1 + r)^{-N}$

$= €15,372,451.03(1.05)^{-9}$

$= €15,372,451.03(0.644609)$

$= €9,909,219.00$

The present value of the pension liability is €9,909,219.00.

Example 14

The Present Value of a Perpetuity

The British government once issued a type of security called a consol bond, which promised to pay a level cash flow indefinitely. If a consol bond paid £100 per year in perpetuity, what would it be worth today if the required rate of return were 5 percent?

Solution:

To answer this question, we can use Equation 13 with the following data:

A=£100r=5%=0.05PV=A/r=£100/0.05=£2,000

$$A = £100$$

$$r = 5\% = 0.05$$

$$PV = A / r$$

$$= £100 / 0.05$$

$$= £2,200$$

The bond would be worth £2,000.

Example 15

The Present Value of a Projected Perpetuity

Consider a level perpetuity of £100 per year with its first payment beginning at t = 5. What is its present value today (at t = 0), given a 5 percent discount rate?

Solution:

First, we find the present value of the perpetuity at t = 4 and then discount that amount back to t = 0. (Recall that a perpetuity or an ordinary annuity has its first payment one period away, explaining the t = 4 index for our present value calculation.)

i. Find the present value of the perpetuity at t = 4:]

A=£100

r=5%

=0.05PV

A=£100r=5%=0.05PV=A/r=£100/0.05=£2,000

PV = A/r

=£100/0.05

==£2,000

ii. Find the present value of the future amount at t = 4. From the perspective of t = 0, the present value of £2,000 can be considered a future value. Now we need to find the present value of a lump sum:

$$FV_N = £2,000 \text{ (the present value at t = 4)}$$

$$r = 5\% = 0.05$$

$$N = 4$$

$$PV = FV_N (1 + r)^{-N}$$

$$= £2,000(1.05)^{-4}$$

$$= £2,000(0.822702)$$

$$= £1,645.40$$

Today's present value of the perpetuity is £1,645.40.

Example 16

The Present Value of an Ordinary Annuity as the Present Value of a Current Minus Projected Perpetuity

Given a 5 percent discount rate, find the present value of a four-year ordinary annuity of £100 per year starting in Year 1 as the difference between the following two level perpetuities:

Perpetuity 1 £100 per year starting in Year 1 (first payment at $t = 1$)

Perpetuity 2 £100 per year starting in Year 5 (first payment at $t = 5$)

Solution:

If we subtract Perpetuity 2 from Perpetuity 1, we are left with an ordinary annuity of £100 per period for four years (payments at $t = 1, 2, 3, 4$). Subtracting the present value of Perpetuity 2 from that of Perpetuity 1, we arrive at the present value of the four-year ordinary annuity:

PV0(Perpetuity1)=£100/0.05=£2,000PV4(Perpetuity2)=£100/0.05=£2,000PV0(Perpetuity2)=£2,000/(1.05)4= £1,645.40PV0(Annuity)=PV0(Perpetuity1)−PV0(Perpetuity2)=£2,000−£1,645.40=£354.60

$$PV_0\text{ (Perpetuity 1)} = £100 / 0.05 = 2,000$$

$$PV_4\text{ (Perpetuity 2)} = £100 / 0.05 = 2,000$$

$$PV_0\text{ (Perpetuity 2)} = £2,000 / (1.05)^4 = 1,645.40$$

$$\begin{aligned} PV_0\text{ (Annuuity)} &= PV_0\text{(Perpetuity)} - PV_0\text{(Perpetuity 2)} \\ &= £2,000 - 1,645.40 \\ &= £354.60 \end{aligned}$$

The four-year ordinary annuity's present value is equal to £2,000 – £1,645.40 = £354.60.

Example 17

Calculating a Growth Rate (1)

Hyundai Steel, the first Korean steelmaker, was established in 1953. Hyundai Steel's sales increased from ₩10,503.0 billion in 2008 to ₩14,146.4 billion in 2012. However, its net profit declined from ₩822.5 billion in 2008 to ₩796.4 billion in 2012. Calculate the following growth rates for Hyundai Steel for the four-year period from the end of 2008 to the end of 2012:

1. Sales growth rate.
2. Net profit growth rate.

Solution to 1:

To solve this problem, we can use Equation 14, $g = (FV_N/PV)^{1/N} - 1$. We denote sales in 2008 as PV and sales in 2012 as FV_4. We can then solve for the growth rate as follows:

$$g = 14,146.4 / \ 10,503.0.4 - 1 = 1.077291 - 1 = 1 = 0.077291 \text{ or about } 7.7\%$$

$$\begin{aligned} g &= \sqrt[4]{w14,146.4 / w10,503.0 - 1} \\ &= \sqrt[4]{1.346891 - 1} \\ &= 1.077291 - 1 \\ &= 0.077291 \text{ or about } 7.7\% \end{aligned}$$

The calculated growth rate of about 7.7 percent a year shows that Hyundai Steel's sales grew substantially during the 2008–2012 period.

Solution to 2:

In this case, we can speak of a positive compound rate of decrease or a negative compound growth rate. Using Equation 14, we find

g = w 796.4 / w822.5.4 - 1 = 0.968267 - 1 = 1 = 0.008029 or about – 0.80%

$$g = \sqrt[4]{w\,796.4 / w822.5.4 - 1}$$

$$= \sqrt[4]{0.968267 - 1}$$

$$= 0.991971 - 1$$

$$= -0.008029 \text{ or about } -0.80\%$$

In contrast to the positive sales growth, the rate of growth in net profit was approximately –0.80 percent during the 2008–2012 period.c

Example 18

Calculating a Growth Rate (2)

Toyota Motor Corporation, one of the largest automakers in the world, had consolidated vehicle sales of 7.35 million units in 2012. This is substantially less than consolidated vehicle sales of 8.52 million units five years earlier in 2007. What was the growth rate in number of vehicles sold by Toyota from 2007 to 2012 ?

Solution:

Using Equation 14, we find

g = 7.35 / 8.52.5 - 1 = 0.8626765 - 1 = 0.970889 – 1 = 0.029111 or about – 2.9%

$$g = \sqrt[5]{7.35 / 8.52} - 1$$

$$= \sqrt[5]{0.862676 - 1}$$

$$= 0.970889 - 1$$

$$= -0.029111 \text{ or about } -2.9\%$$

The rate of growth in vehicles sold was approximately −2.9 percent during the 2007–2012 period. Note that we can also refer to −2.9 percent as the compound annual growth rate because it is the single number that compounds the number of vehicles sold in 2007 forward to the number of vehicles sold in 2012. Table 4 lists the number of vehicles sold operated by Toyota from 2007 to 2012.

Number of Vehicles Sold, 2007–2012   Table 4.

Year	Number of Vehicles Sold (Millions)	$(1 + g)_t$	t
2007	8.52		0
2008	8.91	8.91/8.52 = 1.045775	1
2009	7.57	7.57/8.91 = 0.849607	2
2010	7.24	7.24/7.57 = 0.956407	3
2011	7.31	7.31/7.24 = 1.009669	4
2012	7.35	7.35/7.31 = 1.005472	5

Source: www.toyota.com.

Table 4 also shows 1 plus the one-year growth rate in number of vehicles sold. We can compute the 1 plus five-year cumulative growth in number of vehicles sold from 2007 to 2012 as the product of quantities (1 + one-year growth rate). We arrive at the same result as when we divide the ending number of vehicles sold, 7.35 million, by the beginning number of vehicles sold, 8.52 million:

7.358.52=(8.918.52) (7.578.91) (7.247.57) (7.317.24) (7.357.31) = (1+*g*1) (1+*g*2) (1+*g*3) (1+*g*4) (1+*g*5) 0.862 676 = (1.045775) (0.849607) (0.956407) (1.009669) (1.005472)

$$\frac{7.35}{8.52} = \left(\frac{8.91}{8.52}\right)\left(\frac{7.57}{8.91}\right)\left(\frac{7.24}{7.57}\right)\left(\frac{7.31}{7.24}\right)\left(\frac{7.35}{7.31}\right)$$

$$= (1 + g_1)(1 + g_2)(1 + g_3)(1 + g_4)(1 + g_5)$$

$$0.862676 = (1.045775)(0.849607)(0.956407)(1.009669)(1.005472)$$

The right-hand side of the equation is the product of 1 plus the one-year growth rate in number of vehicles sold for each year. Recall that, using Equation 14, we took the fifth root of 7.35/8.52 = 0.862676. In effect, we were solving for the single value of *g* which, when compounded over five periods, gives the correct product of 1 plus the one-year growth rates.[8]

In conclusion, we do not need to compute intermediate growth rates as in Table 4 to solve for a compound growth rate *g*. Sometimes, however, the intermediate growth rates are interesting or informative. For example, at first (from 2007 to 2008), Toyota Motors increased its number of vehicles sold. We can also analyse the variability in growth rates when we conduct an analysis as in Table 4. Most of the decline in Toyota Motor's sales occurred in 2009. Elsewhere in Toyota Motor's disclosures, the company noted that the substantial decline in vehicle sales in 2009 was due to the steep downturn in the global economy. Sales declined further in 2010 as the market conditions remained difficult. Each of the next two years saw a slight increase in sales.

Example 19

The Number of Annual Compounding Periods Needed for an Investment to Reach a Specific Value

You are interested in determining how long it will take an investment of €10,000,000 to double in value. The current interest rate is 7 percent compounded annually. How many years will it take €10,000,000 to double to €20,000,000?

Solution:

Use Equation 2, $FV_N = PV(1 + r)^N$, to solve for the number of periods, *N*, as follows:

(1+*r*)*N*=FV*N*/PV=2*N*ln(1+*r*)=ln(2)*N*=ln(2)/ln(1+*r*)=ln(2)/ln(1.07)=10.24

$$(1 + r)^N = FV_N / PV = 2$$

$$N \text{ In}(1 + r) = \text{In}(2)$$

$$(n) = \text{In}(2) / \text{In}(1 + r)$$

$$= \text{In}(2) / \text{In}(1.07)$$

$$= 10.24$$

With an interest rate of 7 percent, it will take approximately 10 years for the initial €10,000,000 investment to grow to €20,000,000. Solving for *N* in the expression $(1.07)^N = 2.0$ requires taking the natural logarithm of both sides and using the rule that $\ln(x^N) = N \ln(x)$. Generally, we find that $N = [\ln(FV/PV)]/\ln(1 + r)$. Here, *N* = ln(€20,000,000/ €10,000,000)/ln(1.07) = ln(2)/ln(1.07) = 10.24.[9].

Example 20

Calculating the Size of Payments on a Fixed-Rate Mortgage

You are planning to purchase a $120,000 house by making a down payment of $20,000 and borrowing the remainder with a 30-year fixed-rate mortgage with monthly payments. The first payment is due at *t* = 1. Current mortgage interest rates are quoted at 8 percent with monthly compounding. What will your monthly mortgage payments be ?

Solution:

The bank will determine the mortgage payments such that at the stated periodic interest rate, the present value of the payments will be equal to the amount borrowed (in this case, $100,000). With this fact in mind, we can use Equation 11,

PV = A[1 - 1(1 + r)Nr] $PV = A\left[\dfrac{1 - \dfrac{1}{(1+r)^N}}{r}\right]$ to solve for the annuity amount, A, as

as the present value divided by the present value annuuity factor

to solve for the annuity amount, *A*, as the present value divided by the present value annuity factor:

PV=$100,000*rs*=8%=0.08*m*=12*rs*/*m*=0.08/12=0.006667*N*=30*mN*=12×30=360Presentvalueannuityfactor=1−1[1+(*rs*/*m*)]*mNrs*/*m*=1−1(1.006667)3600.006667=136.283494*A*=PV/Presentvalueannuityfactor=$100,000/136.283494=$733.76

$$
\begin{aligned}
PV &= \$100{,}000 \\
r_s &= 8\% = 0.08 \\
m &= 12 \\
r_s / m &= 0.08/12 = 0.006667 \\
N &= 30 \\
mN &= 12 \times 30 = 360 \\
&\text{Present value annuity factor} \\
&= \frac{1 - \dfrac{1}{[1 + (r_s/m)]^{mN}}}{r_s/m} = \frac{1 - \dfrac{1}{(1.006667)^{360}}}{0.006667} \\
&= 136.283494 \\
A &= PV / \text{Present value annuity factor} \\
&= \$100{,}000 / 136.283494 \\
&= \$733.76
\end{aligned}
$$

The amount borrowed, $100,000, is equivalent to 360 monthly payments of $733.76 with a stated interest rate of 8 percent. The mortgage problem is a relatively straightforward application of finding a level annuity payment.

Example 21

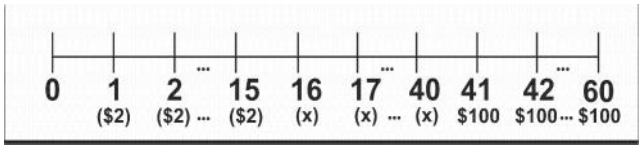
The Projected Annuity Amount Needed to Fund a Future-Annuity Inflow

Jill Grant is 22 years old (at $t = 0$) and is planning for her retirement at age 63 (at $t = 41$). She plans to save $2,000 per year for the next 15 years ($t = 1$ to $t = 15$). She wants to have retirement income of $100,000 per year for 20 years, with the first retirement payment starting at $t = 41$. How much must Grant save each year from $t = 16$ to $t = 40$ in order to achieve her retirement goal? Assume she plans to invest in a diversified stock-and-bond mutual fund that will earn 8 percent per year on average.

Solution:

To help solve this problem, we set up the information on a time line. As Figure 8 shows, Grant will save $2,000 (an outflow) each year for Years 1 to 15. Starting in Year 41, Grant will start to draw retirement income of \$100,000 per year for 20 years. In the time line, the annual savings is recorded in parentheses (\$2) to show that it is an outflow. The problem is to find the savings, recorded as *X*, from Year 16 to Year 40.

Solving for Missing Annuity Payments (in Thousands) Figure 8.



Solving this problem involves satisfying the following relationship: the present value of savings (outflows) equals the present value of retirement income (inflows). We could bring all the dollar amounts to *t* = 40 or to *t* = 15 and solve for *X*.

Let us evaluate all dollar amounts at *t* = 15 (we encourage the reader to repeat the problem by bringing all cash flows to *t* = 40). As of *t* = 15, the first payment of *X* will be one period away (at *t* = 16). Thus we can value the stream of *X*s using the formula for the present value of an ordinary annuity.

This problem involves three series of level cash flows. The basic idea is that the present value of the retirement income must equal the present value of Grant's savings. Our strategy requires the following steps:

1. Find the future value of the savings of \$2,000 per year and index it at *t* = 15. This value tells us how much Grant will have saved.
2. Find the present value of the retirement income at *t* = 15. This value tells us how much Grant needs to meet her retirement goals (as of *t* = 15). Two substeps are necessary. First, calculate the present value of the annuity of \$100,000 per year at *t* = 40. Use the formula for the present value of an annuity. (Note that the present value is indexed at *t* = 40 because the first payment is at *t* = 41.) Next, discount the present value back to *t* = 15 (a total of 25 periods).
3. Now compute the difference between the amount Grant has saved (Step 1) and the amount she needs to meet her retirement goals (Step 2). Her savings from *t* = 16 to *t* = 40 must have a present value equal to the difference between the future value of her savings and the present value of her retirement income.

Our goal is to determine the amount Grant should save in each of the 25 years from *t* = 16 to *t* = 40. We start by bringing the \$2,000 savings to *t* = 15, as follows:

A=\$2,000*r*=8%=0.08*N*=15FV=*A*[(1+*r*)*N*−1*r*]=\$2,000[(1.08)15−10.08]=\$2,000(27.152114)=\$54,304.23

$$A = \$2,000$$

$$r = 8\% = 0.08$$

$$N = 15$$

$$FV = A\left[\frac{(1+r)^{N} - 1}{r}\right]$$

$$= \$2,000\left[\frac{(1.08)^{15} - r}{0.08}\right]$$

$$= \$2,000(27.1521114)$$

$$= \$54,304.23$$

At *t* = 15, Grant's initial savings will have grown to \$54,304.23.

Now we need to know the value of Grant's retirement income at *t* = 15. As stated earlier, computing the retirement present value requires two substeps. First, find the present value at *t* = 40 with the formula in Equation 11; second, discount this present value back to *t* = 15. Now we can find the retirement income present value at *t* = 40:

A=\$100,000*r*=8%=0.08*N*=20PV=*A* 1−1(1+*r*)*Nr* \$100,000 1−1(1.08)200.08 \$100,000(9.818147)=\$981,814.74

$A = \$100,000$

$r = 8\% = 0.08$

$N = 20$

$$FV = A\left[\frac{1 - \frac{1}{(1+r)^N}}{r}\right]$$

$$= \$100,000\left[\frac{1 - \frac{1}{(1.08)^{20}}}{0.08}\right]$$

$= \$100,000(9.818147)$

$= \$100,000(9.818147)$

$= \$981,814.74$

The present value amount is as of $t = 40$, so we must now discount it back as a lump sum to $t = 15$:

FV*N*=$981,814.74*N*=25*r*=8*%*=0.08PV=FV*N*(1+*r*)−*N*=$981,814.74(1.08)−25=$981,814.74(0.146018)=$143,362.53

$FV_N = \$981,814.74$

$N = 25$

$r = 8\% = 0.08$

$PV = FV_N(1+r)^{-N}$

$= \$981,814.74(1.08)-25$

$= \$981,814.74(0.146018)$

$= \$143,362.53$

Now recall that Grant will have saved $54,304.23 by $t = 15$. Therefore, in present value terms, the annuity from $t = 16$ to $t = 40$ must equal the difference between the amount already saved ($54,304.23) and the amount required for retirement ($143,362.53). This amount is equal to $143,362.53 − $54,304.23 = $89,058.30. Therefore, we must now find the annuity payment,*A*, from $t = 16$ to $t = 40$ that has a present value of $89,058.30. We find the annuity payment as follows:

PV=$89,058.30*r*=8*%*=0.08*N*=25Presentvalueannuityfactor=[1−1(1+*r*)*Nr*]=[1−1(1.08)250.08]=10.674776*A*=PV/Presentvalueannuityfactor=$89,058.30/10.674776=$8,342.87

$PV = \$89,058.30$

$r = 8\% = 0.08$

$N = 25$

Present Value annuity factor

$$= \left[\frac{1 - \frac{1}{(1+r)^M}}{r} \right]$$

$$= \left[\frac{1 - \frac{1}{(1.08)^{25}}}{0.08} \right]$$

$= 10.674776$

$A = PV$ / Present value annuity factor

$= \$89,058.30 / 10.674776$

$= \$8,342.87$

Grant will need to increase her savings to \$8,342.87 per year from *t* = 16 to *t* = 40 to meet her retirement goal of having a fund equal to \$981,814.74 after making her last payment at *t* = 40.

QUANTITATIVE METHODS

TIME VALUE OF MONEY

Class Work Questions

Overview —

- **Los a** Interpret interest rates as required rates of return, discount rates, or opportunity costs;
- **Los b** Explain an interest rate as the sum of a real risk-free rate, and premiums that compensate investors against inflation, liquidity, maturity and default risks;
- **Los c** Calculate and interpret the effective annual rate, given the stated annual interest rate and the frequency of compounding;
- **Los d** Solve time value of money problems for different frequencies of compounding;
- **Los e** Calculate and interpret the future value (FV) and present value (PV) of a single sum of money, an ordinary annuity, an annuity due, a perpetuity (PV only), and a series of unequal cash flows;
- **Los f** Demonstrate the use of a time line in modeling and solving time value of money problems;

Los a. Interpret interest rates as required rates of return, discount rates, or opportunity costs;

1. Peter Parker has just inherited some money and wants to set some of it aside for a vacation in Hawaii one year from today. His bank will pay him 5% interest on any funds he deposits. In order to determine how much of the money must be set aside and held for the trip, he should use the 5% as a :

(A) required rate of return

(B) discount rate

(C) opportunity cost

Los b. Explain an interest rate as the sum of a real risk-free rate, and premiums that compensate investors for the frequency of compounding;

2. The real risk-free rate can be thought of as:

(A) approximately the nominal risk-free rate reduced by the expected inflation rate.

(B) exactly the nominal risk-free rate reduced by the expected inflation rate.

(C) approximately the nominal risk-free rate plus the expected inflation rate.

3. T-bill yields can be thought of as:

(A) nominal risk-free rates because they contain an inflation premium.

(B) nominal risk-free rates because they do not contain an inflation premium.

(C) real risk-free rates because they contain an inflation premium.

Los c. Calculate and interpret the effective annual rate, given the stated annual interest rate and the frequency of compounding;

4. A bank pays a stated annual interest rate of 8 percent. What is the effective annual rate using the following types of compounding?

(A) Quarterly.

(B) Monthly.

(C) Continuous.

Los d. Solve time value of money problems for different frequencies of compounding;

5. In 10 years, what is the value of $100 invested today at an interest rate of 8% per year, compounded monthly?

(A) 216

(B) 222

(C) 180

6. If $1,000 is invested at the beginning of the year at an annual rate of 48%, compounded quarterly, what would that investment be worth at the end of the year?

(A) $1,048

(B) $4,798

(C) $1,574

Los e. Calculate and interpret the future value (FV) and present value (PV) of a single sum of money, an ordinary annuity, an annuity due, a perpetuity (PV only), and a series of unequal cash flows;

7. Suppose you own a liquid financial asset that will pay you $100,000 in 10 years from today. Your daughter plans to attend college four years from today, and you want to know what the asset's present value will be at that time. Given an 8 percent discount rate, what will the asset be worth four years from today?

8. Given a 5 percent discount rate, find the present value of a four-year ordinary annuity of £100 per year starting in Year 1 as the difference between the following two level perpetuities:

Perpetuity 1 £100 per year starting in Year 1 (first payment at $t = 1$)

Perpetuity 2 £100 per year starting in Year 5 (first payment at $t = 5$)

9. Emma plans to deposit $500 in her savings account at the end of each quarter for the next 10 years. The interest rate is 10% per year compounded quarterly. After 10 years, her account balance and the total amount of interest that she would have earned are *closest to:*

	Account Balance	Interest Earned
A.	$33,701	$13,254
B.	$33,701	$13,701
C.	$32,504	$13,254

10. Two years from now, a client will receive the first of three annual payments of $20,000 from a small business project. If she can earn 9 percent annually on her investments and plans to retire in six years, how much will the three business project payments be worth at the time of her retirement?

11. To cover the first year's total college tuition payments for his two children, a father will make a $75,000 payment five years from now. How much will he need to invest today to meet his first tuition goal if the investment earns 6 percent annually?

Los f. Demonstrate the use of a time line in modeling and solving time value of money problems;

12. A bank quotes a rate of 5.89 percent with an effective annual rate of 6.05 percent. Does the bank use annual, quarterly, or monthly compounding?

13. Given a discount rate of 11%, which of the following cash flow streams has the *highest* present value?

(A) 12 equal payments of $600 beginning at the end of Year 1.
(B) 10 equal payments of $600 beginning immediately.
(C) 12 equal payments of $550 beginning immediately.

14. Peter must make 6 annual payments of $2,000 each starting at the beginning of Year 5. Given a discount rate of 10%, which of the following methods will enable him to make the required payments:

Method 1: Make 4 equal annual deposits of $1,800 beginning at the end of Year 1.

Method 2: Make 2 equal annual deposits of $4,600 beginning at the end of Year 3.

(A) Method 1 only
(B) Method 2 only
(C) Both the methods

15. Mark Tawson wants to save money for his son's college tuition. His son will start college after 10 years and Tawson expects to make annual payments of $12,500 at the beginning of each year for a 5-year study program with the first payment to be made at the beginning of Year 11. Given a discount rate of 12% the amount that Tawson must deposit at the end of each of the next 10 years is *closest to*:

(A) $2,876
(B) $2,568
(C) $2,658

16. An analyst expects that a company's net sales will double and the company's net income will triple over the next five-year period starting now. Based on the analyst's expectations, which of the following *best* describes the expected compound annual growth?

(A) Net sales will grow 15% annually and net income will grow 25% annually.
(B) Net sales will grow 20% annually and net income will grow 40% annually.
(C) Net sales will grow 25% annually and net income will grow 50% annually.

17. You are considering investing in two different instruments. The first instrument will pay nothing for three years, but then it will pay $20,000 per year for four years. The second instrument will pay $20,000 for three years and $30,000 in the fourth year. All payments are made at year-end. If your required rate of return on these investments is 8 percent annually, what should you be willing to pay for:

(A) The first instrument?

(B) The second instrument (use the formula for a four-year annuity)?

18. What is the expected return from a bond presently trading at $ 1400 and which is expected to pay $ 90 at the end of each year forever starting from t= 5?

Home Work Questions

PRACTICE PROBLEM

1. Alpha research has been conducting investor polls for AY Bank. They have found the most investors are not willing to tie up their money in a 1-year (2-year) CD unless they receive at least 1.0% (1.5%) more than they would on an ordinary savings account. If the savings account rate is 3%, and the bank wants to raise funds with 2-year CDs, the yield must be at least:

(A) 4.5%, and this represents a required rate of return.

(B) 4.0%, and this represents a required rate of return.

(C) 4.5%, and this represents a discount rate.

2. Miss Marry has just inherited some money and wants to set some of it aside for a vacation in Hawaii one year from today. Her bank will pay him 5% interest on any funds he deposits. In order to determine how much of the money must be set aside and held for the trip, he should use the 5% as a:

(A) required rate of return.

(B) discount rate.

(C) opportunity cost.

3. Which one of the following statements *best* describes the components of the required interest rate on a security?

(A) The nominal risk-free rate, the expected inflation rate, the default risk premium, a liquidity premium and a premium to reflect the risk associated with the maturity of the security.

(B) The real risk-free rate, the default risk premium, a liquidity premium and a premium to reflect the risk associated with the maturity of the security.

(C) The real risk-free rate, the expected inflation rate, the default risk premium, a liquidity premium and a premium to reflect the risk associated with the maturity of the security.

4. T-bill yields can be thought of as:

(A) nominal risk-free rates because they contain an inflation premium.

(B) nominal risk-free rates because they do not contain an inflation premium.

(C) real risk-free rates because they contain an inflation premium.

5. Which of the following is least likely a premium that compensates investors for bearing risk?

(A) Default risk premium

(B) Maturity premium

(C) Payback premium

6. Stella invested $3,000 in an account with an interest rate of 15% compounded continuously. After 5 years, the value of her investment will be closest to:

(A) $6,351

(B) $6,030

(C) $6,240

7. A local bank offers an account that pays 8%, compounded quarterly, for any deposits of $10,000 or more that are left in the account for a period of 5 years. The effective annual rate of interest on this account is:

(A) 4.65%.

(B) 9.01%.

(C) 8.24%.

8. Which of the following is the *most* accurate statement about stated and effective annual interest rates?

A) The stated rate adjusts for the frequency of compounding.

B) So long as interest is compounded more than once a year, the stated annual rate will always be more than the effective rate.

C) The stated annual interest rate is used to find the effective annual rate.

9. A major brokerage firm is currently selling an investment product that offers an 8% rate of return, compounded monthly. Based on this information, it follows that this investment has:

A) an effective annual rate of 8.00%.

B) a periodic interest rate of 0.667%.

C) a stated rate of 0.830%.

10. Use a stated rate of 9% compounded periodically to answer the following three questions. Select the choice that is the *closest* to the correct answer.

(I) The semi-annual effective rate is:

A) 9.20%.

B) 9.00%

C) 9.31%

(II) The quarterly effective rate is:

A) 9.40%

B) 9.31%

C) 9.00%

(III) The continuously compounded rate is:

A) 9.67%

B) 9.42%

C) 9.20%

11. What is the effective rate of return on an investment that generates a return of 12%, compounded quarterly?

(A) 14.34%.

(B) 12.00%.

(C) 12.55%.

12. Andrew Watson wants to deposit $10,000 in a bank certificate of deposit (CD). Wallace is considering the following banks:

Bank X offers 5.85% annual interest compounded annually.

Bank Y offers 5.75% annual interest rate compounded monthly.

Bank Z offers 5.70% annual interest compounded daily.

Which bank offers the highest effective interest rate ?

(A) Bank X, 5.85%.

(B) Bank Y, 5.90%.

(C) Bank Z, 5.87%.

13. An investor puts $3,000 in a bank account that offers an annual interest rate of 11% compounded daily. The account balance at the end of Year 1 will be closest to:

(A) $3,348.78

(B) $3,000.10

(C) $2,997.44

14. Given: an 11% annual rate compounded quarterly for 2 years; compute the future value of $8,000 today.

(A) $8,962.

(B) $9,857.

(C) $9,939.

15. Given: $1,000 investment, compounded monthly at 12% find the future value after one year.

(A) $1,126.83.

(B) $1,121.35.

(C) $1,120.00.

16. What is the maximum price an investor should be willing to pay (today) for a 10 year annuity that will generate $500 per quarter (such payments to be made at the end of each quarter), given he wants to earn 12%, compounded quarterly?

(A) $6,440.

(B) $11,557.

(C) $11,300.

17. An investor invested $10,000 into an account five years ago. Today, the account value is $18,682. What is the investor's annual rate of return on a continuously compounded basis?

(A) 11.33%.

(B) 12.50%.

(C) 13.31%.

18. Jones needs to accumulate $2,000 in 18 months. If she can earn 6% at the bank, compounded quarterly, how much must she deposit today?

(A) $1,832.61.

(B) $1,840.45.

(C) $1,829.08.

19. Mr. Whales invests $2,000 each year, starting one year from now, in a retirement account. If the investments earn 8% or 10% annually over 30 years, the amount Whales will accumulate is closest to:

	8%	10%
(A)	$225,000	$360,000
(B)	$245,000	$360,000
(C)	$225,000	$330,000

20. If 10 equal annual deposits of $1,000 are made into an investment account earning 9% starting today, how much will you have in 20 years?

(A) $39,204.

(B) $35,967.

(C) $42,165.

21. An annuity will pay eight annual payments of $100, with the first payment to be received three years from now. If the interest rate is 12% per year, what is the present value of this annuity? The present value of:

(A) a lump sum discounted for 3 years, where the lump sum is the present value of an ordinary annuity of 8 periods at 12%.

(B) an ordinary annuity of 8 periods at 12%.

(C) a lump sum discounted for 2 years, where the lump sum is the present value of an ordinary annuity of 8 periods at 12%.

22. An investor deposits $4,000 in an account that pays 7.5%, compounded annually. How much will this investment be worth after 12 years?

(A) $5,850.

(B) $9,527.

(C) $9,358.

23. An investment offers $100 per year forever. If Cooper Lopez required rate of return on this investment is 10%, how much is this investment worth to him?

(A) $1,000.

(B) $10,000.

(C) $500.

24. What is the present value of a 12-year annuity due that pays $5,000 per year, given a discount rate of 7.5%?

(A) $41,577.

(B) $36,577.

(C) $38,676.

25. How much would the following income stream be worth assuming a 12% discount rate?

- $100 received today.
- $200 received 1 year from today.
- $400 received 2 years from today.
- $300 received 3 years from today.

(A) $810.98.

(B) $721.32.

(C) $1,112.44.

26. Concerning an ordinary annuity and an annuity due with the same payments and positive interest rate, which of the following statements is most accurate?

(A) The present value of the ordinary annuity is greater than an annuity due.

(B) The present value of the ordinary annuity is less than an annuity due.

(C) There is no relationship.

27. Consider a 10-year annuity that promises to pay out $10,000 per year; given this is an ordinary annuity and that an investor can earn 10% on her money, the future value of this annuity, at the end of 10 years, would be:

(A) $175,312.

(B) $159,374.

(C) $110.000.

28. An investor will receive an annuity of $5,000 a year for seven years. The first payment is to be received 5 years from today. If the annual interest rate is 11.5%, what is the present value of the annuity?

(A) $13,453.

(B) $23,185.

(C) $15,000.

29. What is the present value of a 10-year, $100 annual annuity due if interest rates are 0%?

(A) $900.

(B) No solution.

(C) $1,000.

30. Compute the present value of a perpetuity with $100 payments beginning four years from now. Assume the appropriate annual interest rate is 10%.

(A) $683.

(B) $1000.

(C) $751.

31. Caron Industries has a preferred stock outstanding that pays (fixed) annual dividends of $3.75 a share. If an investor wants to earn a rate of return of 8.5%, how much should he be willing to pay for a share of Caron preferred stock?

(A) $31.88.

(B) $42.10.

(C) $44.12.

32. Assuming a discount rate of 10%, which stream of annual payments has the highest present value?

(A) Year1- $110, Year2- $20, Year3- $10, Year4- $5

(B) Year1- $20, Year2- $5, Year3- $20, Year4- $110

(C) Year1- $(100), Year2- $(100), Year3- $(100), Year4- $500

33. If $2,000 a year is invested at the end of each of the next 45 years in a retirement account yielding 8.5%, how much will an investor have at retirement 45 years from today?

(A) $100,135.

(B) $90,106.

(C) $901,060.

34. James purchases a 10-year, $1,000 par value bond that pays annual coupons of $100. If the market rate of interest is 12%, what is the current market value of the bond?

(A) $1,124.

(B) $887.

(C) $950.

35. Given investors require an annual return of 12.5%, a perpetual bond (i.e., a bond with no maturity/due date) that pays $87.50 a year in interest should be valued at:

(A) $70.

(B) $1,093.

(C) $700.

36. An investor deposits $10,000 in a bank account paying 5% interest compounded annually. Rounded to the nearest dollar, in 5 years the investor will have:

(A) $12,500.

(B) $12,763.

(C) $10,210.

37. What will $10,000 become in 5 years if the annual interest rate is 8%, compounded monthly?

(A) $14,693.28.

(B) $14,802.44.

(C) $14,898.46.

38. A $500 investment offers a 7.5% annual rate of return. How much will it be worth in four years?

(A) $892.

(B) $668.

(C) $650.

39. A bank offers a certificate of deposit (CD) that earns 5.0% compounded quarterly for three and one half years. If a depositor places $5,000 on deposit, what will be the value of the account at maturity?

(A) $5,931.06.

(B) $5,949.77.

(C) $5,875.00.

40. Nelson Fortin, neurosurgeon at a large U.S. university, was recently granted permission to take an 18-month sabbatical that will begin one year from today. During the sabbatical, Brunswick will need $2,500 at the beginning of each month for living expenses that month. Her financial planner estimates that she will earn an annual rate of 9% over the next year on any money she saves. The annual rate of return during her sabbatical term will likely increase to 10%. At the end of each month during the year before the sabbatical, Brunswick should save approximately:

(A) $3,356.

(B) $3,505.

(C) $3,330.

41. An individual borrows $200,000 to buy a house with a 30-year mortgage requiring payments to be made at the end of each month. The interest rate is 8%, compounded monthly. What is the monthly mortgage payment?

(A) $2,142.39.

(B) $1,467.53.

(C) $1,480.46.

42. A recent ad for a local bank includes the statement that if a person invests $500 at the beginning of each month for 35 years, they could have $1,000,000 for retirement. Assuming monthly compounding, what annual interest rate is implied in this statement?

(A) 7.411%.

(B) 7.625%.

(C) 6.988%.

43. Which of the following statements about compounding and interest rates is least accurate?

(A) Present values and discount rates move in opposite directions.

(B) On monthly compounded loans, the effective annual rate (EAR) will exceed the annual percentage rate (APR).

(C) All else equal, the longer the term of a loan, the lower will be the total interest you pay.

44. Kelly Diaz, hedge fund manager and avid downhill skier, was recently granted permission to take a 4 month sabbatical. During the sabbatical, (scheduled to start in 11 months), Diaz will ski at approximately 12 resorts located in the Austrian, Italian, and Swiss Alps. Diaz estimates that she will need $6,000 at the beginning of each month for expenses that month. (She has already financed her initial travel and equipment costs.) Her financial planner estimates that she will earn an annual rate of 8.5% during her savings period and an annual rate of return during her sabbatical of 9.5%. How much does she need to put in her savings account at the end of each month for the next 11 months to ensure the cash flow she needs over her sabbatical? Each month, Diaz should save approximately:

(A) $2,065.

(B) $2,070.

(C) $2,080.

45. Hakib Ali and Davis Garcia borrowed $15,000 to help finance their wedding and reception. The annual payment loan carries a term of seven years and an 11% interest rate. Respectively, the amount of the first payment that is interest and the amount of the second payment that is principal are approximately:

(A) $1,650; $1,702.

(B) $1,468; $1,702.

(C) $1,650; $1,468.

46. Bell Gagne wants to give his son a new car for his graduation. If the cost of the car is $15,000 and Bell finances 80% of the value of the car for 36 months at 8% annual interest, his monthly payments will be:

(A) $376.

(B) $413.

(C) $289.

47. Lui Shen wants to have $1.5 million in a retirement fund when he retires in 30 years. If Shen can earn a 9% rate of return on her investments, approximately how much money must he invest at the end of each of the next 30 years in order to reach his goal?

(A) $50,000.

(B) $11,005.

(C) $28,725.

48. Ella wants to start college after 5 years for which she wants to accumulate money by making annual deposits in her bank account beginning at the end of Year 1. She estimates that she will have to make a payment of $8,000 at the beginning of each year through the 4-year study program with the first payment to be made at the beginning of Year 6. Given a discount rate of 11%, the amount that must be deposited at the end of each year for the next 5 years to satisfy the eventual payment obligations is closest to:

(A) $3,985
(B) $6,400
(C) $4,424

49. Davis Fortin wants to save money for his son's college tuition. His son will start college after 10 years and Fortin expects to make annual payments of $12,500 at the beginning of each year for a 5-year study program with the first payment to be made at the beginning of Year 11. Given a discount rate of 12% the amount that Fortin must deposit at the end of each of the next 10 years is closest to:
(A) $2,876
(B) $2,568
(C) $2,658

50. Selena is 30 years old and wants to retire in 20 years at the age of 50. She expects to earn 12% on her savings prior to retirement and 8% thereafter. Martha wants to be able to withdraw $18,000 every year at the beginning of each year for 30 years from the age of 50 to 80. The amount that she must deposit at the end of each year for the next 20 years is closest to:
(A) $2,812
(B) $2,724
(C) $3,037

51. Howard borrowed $12,000 to start his own business. The loan must be repaid through four equal end-of-year payments and carries an interest rate of 8%. The principal component of the second payment is closest to:
(A) $2,876
(B) $2,663
(C) $3,623

52. Given a discount rate of 11%, which of the following cash flow streams has the highest present value?
(A) 12 equal payments of $600 beginning at the end of Year 1.
(B) 10 equal payments of $600 beginning immediately.
(C) 12 equal payments of $550 beginning immediately.

SOLUTION

1. Answer A
Since we are taking the view of the minimum amount required to induce investors to lend funds to the bank, this is best described as a required rate of return. Based upon the numerical information, the rate must be 4.5% (= 3.0 + 1.5).

2. Answer B
She needs to figure out how much the trip will cost in one year, and use the 5% as a discount rate to convert the future cost to a present value. Thus, in this context the rate is best viewed as a discount rate.

3. Answer C
The required interest rate on a security is made up of the nominal rate which is in turn made up of the real risk-free rate plus the expected inflation rate. It should also contain a liquidity premium as well as a premium related to the maturity of the security.

4. Answer A
T-bills are government issued securities and are therefore considered to be default risk free. More precisely, they are nominal risk-free rates rather than real risk-free rates since they contain a premium for expected inflation.

5. Answer C

Default risk premium, inflation premium, liquidity premium, and maturity premium are examples of premiums that compensate investors for taking on different kinds of risks.

6. **Answer: A**

   $FV = PVe^{rt}$

   PV = −$3,000; r = 0.15; t = 5

   FV = $6,351

7. **Answer C**

   $(1 + \text{periodic rate})^{m} - 1 = (1.02)^{4} - 1 = 8.24\%$.

8. **Answer C**

   The effective annual rate, not the stated rate, adjusts for the frequency of compounding. The nominal, stated, and stated annual rates are all the same thing.

9. **Answer B**

   Periodic rate = 8.0 / 12 = 0.667. Stated rate is 8.0% and effective rate is 8.30%.

10. **(i) Answer A**

    First, we need to calculate the periodic rate, or 0.09 / 2 = 0.045.

    Then, the effective semi-annual rate = $(1 + 0.045)^{2} - 1 = 0.09203$, or 9.20%.

    (ii) Answer B

    First, we need to calculate the periodic rate, or 0.09 / 4 = 0.0225.

    Then, the effective annual rate = $(1 + 0.0225)^{4} - 1 = 0.09308$, or 9.31%.

    (iii) Answer B

    The continuously compounded rate = $e^{r} - 1 = e^{0.09} - 1 = 0.09417$, or 9.42%.

    Calculator Keystrokes for e^{t}: Using the TI BA, enter [0.09] [2nd] [e^{x}] (this is the key with LN on the face of the button). On the HP, enter [0.09] [g] [e^{x}] (this key is located in blue on the key with 1/x in white print).

11. **Answer C**

    $(1 + 0.12 / 4)^{4} - 1 = 1.1255 - 1 = 0.1255$.

12. **Answer B**

    Effective interest rates:

    Bank X = 5.85 (already annual compounding)

    Bank Y, nominal = 5.75; C/Y = 12; effective = 5.90

    Bank Z, nominal = 5.70, C/Y = 365; effective = 5.87

    Hence Bank Y has the highest effective interest rate.

13. **Answer: A**

    PV = −$3,000, I/Y = 11/365; N = 365; CPT→ FV; FV = $3,348.78.

14. **Answer C**

    Divide the interest rate by the number of compound periods and multiply the number of years by the number of compound periods. I = 11 / 4 = 2.75; N = (2)(4) = 8; PV = 8,000.

15. **Answer A**

    Divide the interest rate by the number of compound periods and multiply the number of years by the number of compound periods. I = 12 / 12 = 1; N = (1)(12) = 12; PV = 1,000.

16. **Answer B**

    Using a financial calculator: N = 10 × 4 = 40; I/Y = 12 / 4 = 3; PMT = -500; FV = 0; CPT → PV = 11,557.

17. **Answer B**
    ln(18,682/10,000) = 0.6250/5 = 12.50%
    or
    $(18{,}682/10{,}000)^{1/5}$ = 1.133143
    ln(1.133143) = 12.4995%
18. **Answer C**
    Each quarter of a year is comprised of 3 months thus N = 18 / 3 = 6; I/Y = 6 / 4 = 1.5; PMT = 0; FV = 2,000; CPT → PV = $1,829.08.
19. **Answer C**
    N = 30; I/Y = 8; PMT = -2,000; PV = 0; CPT FV = 226,566.42
    N = 30; I/Y = 10; PMT = -2,000; PV = 0; CPT FV = 328,988.05
20. **Answer A**
    Switch to BGN mode. PMT = 1,000; N = 10, I/Y = 9, PV = 0; CPT → FV = 16,560.29. Remember the answer will be one year after the last payment in annuity due FV problems. Now PV_{10} = 16,560.29; N = 10; I/Y = 9; PMT = 0; CPT → FV = 39,204.23. Switch back to END mode.
21. **Answer C**
    The PV of an ordinary annuity (calculation END mode) gives the value of the payments one period before the first payment, which is a time = 2 value here. To get a time = 0 value, this value must be discounted for two periods (years).
22. **Answer B**
    N = 12; I/Y = 7.5; PV = -4,000; PMT = 0; CPT → FV = $9,527.
23. **Answer A**
    For a perpetuity, PV = PMT ÷ I = 100 ÷ 0.10 = 1,000.
24. **Answer A**
    Using your calculator: N = 11; I/Y = 7.5; PMT = -5,000; FV = 0; CPT → PV = 36,577 + 5,000 = $41,577. Or set your calculator to BGN mode and N = 12; I/Y = 7.5; PMT = -5,000; FV = 0; CPT → PV = $41,577.
25. **Answer A**
    We have,

N	**I**	**FV**	**PV**
0	12	100	100.00
1	12	200	178.57
2	12	400	318.88
3	12	300	213.53
			810.98

26. **Answer B**
    With a positive interest rate, the present value of an ordinary annuity is less than the present value of an annuity due. The first cash flow in an annuity due is at the beginning of the period, while in an ordinary annuity, the first cash flow occurs at the end of the period. Therefore, each cash flow of the ordinary annuity is discounted one period more.
27. **Answer B**
    N = 10; I/Y = 10; PMT = -10,000; PV = 0; CPT → FV = $159,374.
28. **Answer C**
    With PMT = 5,000; N = 7; I/Y = 11.5; value (at t = 4) = 23,185.175. Therefore, PV (at t = 0) = 23,185.175 / $(1.115)^4$ = $15,000.68.

29. Answer C

When I/Y = 0 you just sum up the numbers since there is no interest earned.

30. Answer C

Compute the present value of the perpetuity at (t = 3). Recall, the present value of a perpetuity or annuity is valued one period before the first payment. So, the present value at t = 3 is 100 / 0.10 = 1,000. Now it is necessary to discount this lump sum to t = 0. Therefore, present value at t = 0 is 1,000 / $(1.10)^3$ = 751.

31. Answer C

PV = 3.75 / 0.085 = $44.12.

32. Answer A

This is an intuition question. The two cash flow streams that contain the $110 payment have the same total cash flow but the correct answer is the one where the $110 occurs earlier. The cash flow stream that has the $500 that occurs four years hence is overwhelmed by the large negative flows that precede it.

33. Answer C

N = 45; PMT = 2,000; PV = 0; I/Y = 8.5%; CPT → FV = $901,060.79.

34. Answer B

Note that bond problems are just mixed annuity problems. You can solve bond problems directly with your financial calculator using all five of the main TVM keys at once. For bond-types of problems the bond's price (PV) will be negative, while the coupon payment (PMT) and par value (FV) will be positive.

N = 10; I/Y = 12; FV = 1,000; PMT = 100; CPT → PV = 886.99.

35. Answer C

87.50 ÷ 0.125 = $700.

36. Answer A

PV = 10,000; I/Y = 5; N = 5; CPT → FV = 12,763.

or: $10{,}000(1.05)^5$ = 12,763.

37. Answer C

$FV_{(t=5)}$ = $10,000 × $(1 + 0.08 / 12)^{60}$ = $14,898.46

N = 60 (12 × 5); PV = -$10,000; I/Y = 0.66667 (8% / 12months); CPT → FV = $14,898.46

38. Answer B

N = 4; I/Y = 7.5; PV = 500; PMT = 0; CPT → FV = 667.73.

or: $500(1.075)^4$ = 667.73

39. Answer B

The value of the account at maturity will be: $5,000 × $(1 + 0.05 / 4)^{(3.5 \times 4)}$ = $5.949.77; or with a financial calculator: N = 3 years × 4 quarters/year + 2 = 14 periods; I = 5% / 4 quarters/year = 1.25; PV = $5,000; PMT = 0; CPT → FV = $5,949.77.

40. Answer B

This is a two-step problem. First, we need to calculate the present value of the amount she needs over her sabbatical. (This amount will be in the form of an annuity due since she requires the payment at the beginning of the month.) Then, we will use future value formulas to determine how much she needs to save each month (ordinary annuity).

Step 1: *Calculate present value of amount required during the sabbatical*

Using a financial calculator:

Set to BEGIN Mode, then N = 12 × 1.5 = 18; I/Y = 10 / 12 = 0.8333; PMT = 2,500; FV = 0; CPT → PV = 41,974

Step 2: *Calculate amount to save each month*

Make sure the calculator is set to END mode,

then N = 12; I/Y = 9 / 12 = 0.75; PV = 0; FV = 41,974; CPT → PMT = -3,356

41. Answer B

With PV = 200,000; N = 30 × 12 = 360; I/Y = 8/12; CPT → PMT = $1,467.53.

42. Answer A

Solve for an annuity due with a future value of $1,000,000, a number of periods equal to (35 × 12) = 420, payments = -500, and present value = 0. Solve for i. i = 0.61761 × 12 = 7.411% stated annually. Don't forget to set your calculator for payments at the beginning of the periods. If you don't, you'll get 7.437%.

43. Answer C

Since the proportion of each payment going toward the principal decreases as the original loan maturity increases, the total dollars interest paid over the life of the loan also increases.

44. Answer C

This is a two-step problem. First, we need to calculate the present value of the amount she needs over her sabbatical. (This amount will be in the form of an annuity due since she requires the payment at the beginning of the month.) Then, we will use future value formulas to determine how much she needs to save each month.

Step 1: *Calculate present value of amount required during the sabbatical*

Using a financial calculator: Set to BEGIN Mode, then N = 4; I/Y = 9.5 / 12 = 0.79167; PMT = 6,000; FV = 0; CPT → PV = -23,719.

Step 2: *Calculate amount to save each month*

Using a financial calculator: Make sure it is set to END mode, then N = 11; I/Y = 8.5 / 12.0 = 0.70833; PV = 0; FV = 23,719; CPT → PMT= -2,081, or approximately $2,080.

45. Answer A

Step 1: *Calculate the annual payment.*

Using a financial calculator (remember to clear your registers): PV = 15,000; FV = 0; I/Y = 11; N = 7; PMT = $3,183

Step 2: *Calculate the portion of the first payment that is interest.*

Interest_1 = Principal × Interest rate = (15,000 × 0.11) = 1,650

Step 3: *Calculate the portion of the second payment that is principal.*

Principal_1 = Payment − Interest_1 = 3,183 − 1,650 = 1,533 (interest calculation is from Step 2)

Interest_2 = Principal remaining × Interest rate = [(15,000 − 1.533) × 0.11] = 1,481

Principal_2 = Payment − Interest_1 = 3,183 − 1,481 = 1,702

46. Answer A

PV = 0.8 × 15,000 = -12,000; N = 36; I = 8/12 = 0.667; CPT → PMT = 376.

47. Answer B

Using a financial calculator: N = 30; I/Y = 9; FV = -1,500,000; PV = 0; CPT → PMT = 11,004.52.

48. Answer: C

With the calculator in BGN mode:

N = 4; I/Y = 11; PMT = −$8,000; CPT PV; PV → $27,549.72 (PV of obligations at the end of Year 5/ Beginning of Year 6)

With the calculator in END mode:

N = 5; I/Y = 11; FV = −$27,549.72 (required amount); CPT PMT; PMT → $4,423.67

49. Answer: A

With the calculator in BGN mode:

N = 5; I/Y = 12; PMT = −$12,500; CPT PV; PV → $50,466.867

With the calculator in END mode:

N = 10; I/Y = 12; FV = −$50,466.867; CPT PMT; PMT → $2,875.81

50. Answer: C

With the calculator in BGN mode:

N = 30; I/Y = 8; PMT = −$18,000; CPT PV; PV → $218,851.31 (Value of withdrawals during retirement)

With the calculator in END mode:

N = 20; I/Y = 12; FV = −$218,851.31; CPT PMT; PMT → $3,037.39

51. Answer: A

N = 4; I/Y = 8; PV = $12,000; CPT PMT; PMT = −$3,623.05

Interest component of the first payment = $12,000 × 0.08 = $960

Principal component of the first payment = $3,623.05 – $960 = $2,663.05

Interest component of the second payment = ($12,000 – $2,663.05) × 0.08 = $746.96

Principal component of the second payment = $3,623.05 – $746.96 = $2,876.09

52. Answer: C

N = 12; I/Y = 11; PMT = −$600; CPT PV; PV → $3,895.414

Set the calculator to BGN mode.

N = 10; I/Y = 11; PMT = −$600; CPT PV; PV → $3,922.23

N = 12; I/Y = 11; PMT = −$550; CPT PV; PV → $3,963.58

<u>THE TIME VALUE OF MONEY</u>

SUMMARY:

In this reading, we have explored a foundation topic in investment mathematics, the time value of money. We have developed and reviewed the following concepts for use in financial applications:

- ✓ The interest rate, r, is the required rate of return; r is also called the discount rate or opportunity cost.

- ✓ An interest rate can be viewed as the sum of the real risk-free interest rate and a set of premiums that compensate lenders for risk: an inflation premium, a default risk premium, a liquidity premium, and a maturity premium.

- ✓ The future value, FV, is the present value, PV, times the future value factor, $(1 + r)N$.

- ✓ The interest rate, r, makes current and future currency amounts equivalent based on their time value.

- ✓ The stated annual interest rate is a quoted interest rate that does not account for compounding within the year.

- ✓ The periodic rate is the quoted interest rate per period; it equals the stated annual interest rate divided by the number of compounding periods per year.

- ✓ The effective annual rate is the amount by which a unit of currency will grow in a year with interest on interest included.

- ✓ An annuity is a finite set of level sequential cash flows.

- ✓ There are two types of annuities, the annuity due and the ordinary annuity. The annuity due has a first cash flow that occurs immediately; the ordinary annuity has a first cash flow that occurs one period from the present (indexed at $t = 1$).

- ✓ On a time line, we can index the present as 0 and then display equally spaced hash marks to represent a number of periods into the future. This representation allows us to index how many periods away each cash flow will be paid.

- ✓ Annuities may be handled in a similar fashion as single payments if we use annuity factors instead of single-payment factors.

- ✓ The present value, PV, is the future value, FV, times the present value factor, $(1 + r)-N$.

- ✓ The present value of a perpetuity is A/r, where A is the periodic payment to be received forever.

- ✓ It is possible to calculate an unknown variable, given the other relevant variables in time value of money problems.

- ✓ The cash flow additively principle can be used to solve problems with uneven cash flows by combining single payments and annuities.